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Researches into the Mathematical Principles of the Theory of Wealth

Augustin Cournot

This article is a reprint of Chapters VII and IX of Augustin Cournot, *Researches into the Mathematical Principles of the Theory of Wealth* (N.T. Bacon trans.) (New York: Augustus M. Kelley, 1971) (1838).

Chapter VII: Of the Competition of Producers

43. Every one has a vague idea of the effects of competition. Theory should have attempted to render this idea more precise; and yet, for lack of regarding the question from the proper point of view, and for want of recourse to symbols (of which the use in this connection becomes indispensable), economic writers have not in the least improved on proper notions in this respect. These notions have remained as ill-defined and ill-applied in their works, as in popular language.

To make the abstract idea of monopoly comprehensible, we imagined one spring and one proprietor. Let us now imagine two proprietors and two springs of which the qualities are identical, and which, on account of their similar positions, supply the same market in competition. In this case the price is necessarily the same for each proprietor. If p is this price, $D = F(p)$ the total sales, D_1 the sales from spring (1) and D_2 the sales from the spring (2), then $D_1 + D_2 = D$. If, to begin with, we neglect the cost or production, the respective incomes of the proprietors will be pD_1 and pD_2; and *each of them independently* will seek to make this income as large as possible.

We say *each independently*, and this restriction is very essential, as will soon appear; for if they should come to an agreement so as to obtain for each the greatest possible income, the results would be entirely different, and would not differ, so far as consumers are concerned, from those obtained in treating of a monopoly.

Instead of adopting $D = F(p)$ as before, in this case it will be convenient to adopt the inverse notation $p = f(D)$; and then the profits of proprietors (1) and (2) will be respectively expressed by

$$D_1 \times f(D_1 + D_2), \text{ and } D_2 \times f(D_1 + D_2),$$

i.e. by functions into each of which enter two variables, D_1 and D_2.

Proprietor (1) can have no direct influence on the determination of D_2: all that he can do, when D_2 has been determined by proprietor (2), is to choose for D_1 the value which is best for him. This he will be able to accomplish by properly adjusting his price, except as proprietor (2), who, seeing himself forced to accept this price and this value of D_1, may adopt a new value for D_2, more favourable to his interests than the preceding one.

Analytically this is equivalent to saying that D_1 will be determined in terms of D_2 by the condition

$$\frac{d[D_1 f(D_1 + D_2)]}{dD_1} = 0,$$

and that D_2 will be determined in terms of D_1 by the analogous condition

$$\frac{d[D_2 f(D_1 + D_2)]}{dD_2} = 0,$$

whence it follows that the final values of D_1 and D_2, and consequently of D and of p, will be determined by the system of equations

(1) $$f(D_1 + D_2) + D_1 f'(D_1 + D_2) = 0,$$

(2) $$f(D_1 + D_2) + D_2 f'(D_1 + D_2) = 0.$$

Let us suppose the curve $m_1 n_1$ (Fig. 2) to be the plot of equation (1), and the curve $m_2 n_2$ that of equation (2), the variables D_1 and D_2 being represented by rectangular coördinates.

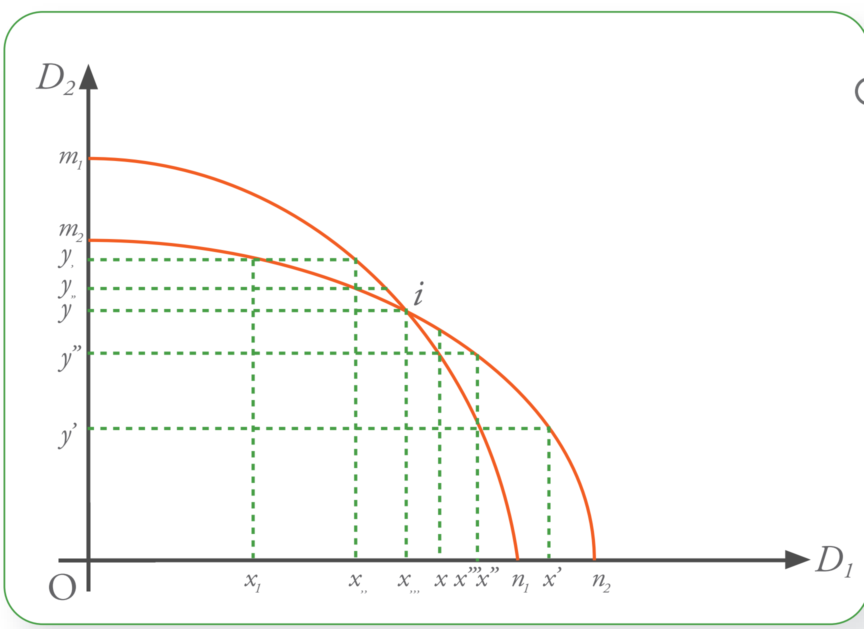


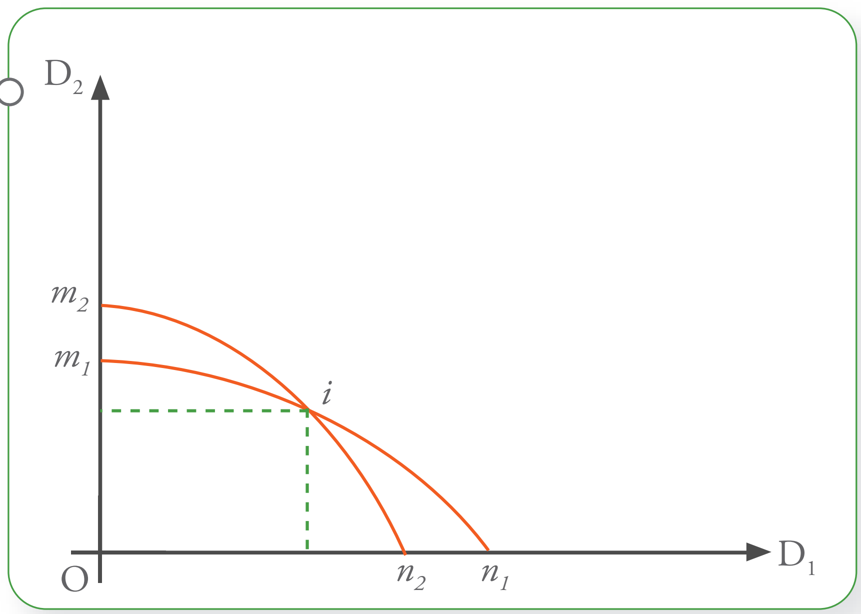
Figure 2

If proprietor (1) should adopt for D_1 a value represented by ox_1, proprietor (2) would adopt for D_2 the value oy_1, which, for the supposed value of D_1, would give him the greatest profit. But then, for the same reason, producer (1) ought to adopt for D_1 the value ox_{11}, which gives the maximum profit when D_2 has the value oy_1. This would bring producer (2) to the value oy_{11} for D_2, and so forth; from which it is evident that an equilibrium can only be established where the coördinates ox and oy of the point of intersection i represent the values of D_1 and D_2. The same construction repeated on a point of the figure on the other side of the point i leads to symmetrical results.

The state of equilibrium corresponding to the system of values ox and oy is therefore *stable; i.e.* if either of the producers, misled as to his true interest, leaves it temporarily, he will be brought back to it by a series of reactions, constantly declining in amplitude, and of which the dotted lines of the figure give a representation by their arrangement in steps.

The preceding construction assumes that $om_1 > om_2$ and $on_1 < on_2$: the results would be diametrically opposite if these inequalities should change sign, and if the curves m_1n_1 and m_2n_2 should assume the disposition represented by Fig. 3.

Figure 3



The coördinates of the point i, where the two curves intersect, would then cease to correspond to a state of stable equilibrium. But it is easy to prove that such a disposition of the curves is inadmissible. In fact, if $D_1 = 0$, equations (1) and (2) reduce, the first to

$$f(D_2) = 0,$$

and the second to

$$f(D_2) + D_2 f'(D_2) = 0.$$

The value of D_2 derived from the first would correspond to $p = 0$; the value of D_2 derived from the second corresponds to a value of p which would make the product pD_2 a maximum. Therefore the first root is necessarily greater than the second, or $om_1 > om_2$, and for the same reason $on_2 > on_1$.

44. From equations **(1)** and **(2)** we derive first $D_1 = D_2$ (which ought to be the case, as the springs are supposed to be similar and similarly situated), and then by addition:

$$2f(D) + Df'(D) = 0,$$

an equation which can be transformed into

(3)
$$D + 2p \frac{dD}{dp} = 0,$$

whereas, if the two springs had belonged to the same property, or if the two proprietors *had come to an understanding*, the value of p would have been determined by the equation

(4)
$$D + p \frac{dD}{dp} = 0,$$

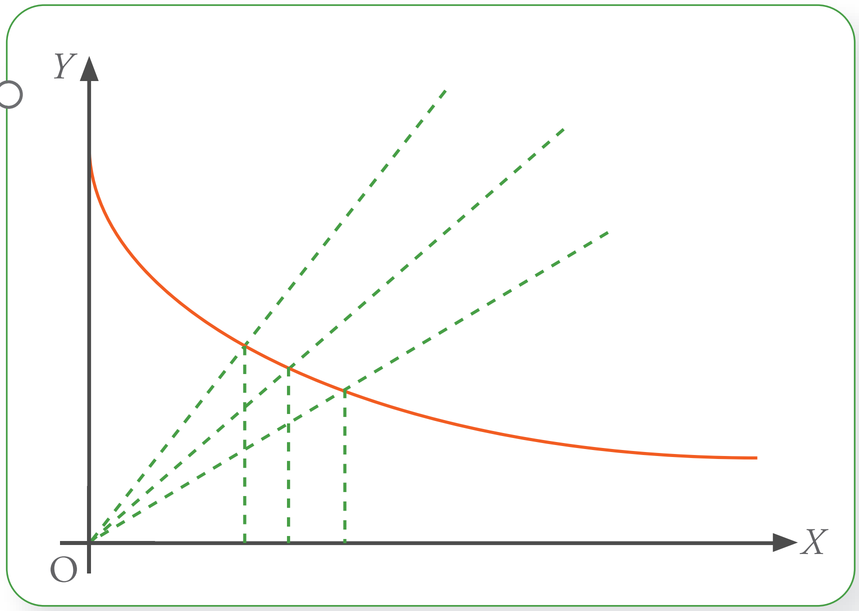
and would have rendered the total income Dp a *maximum*, and consequently would have assigned to each of the producers a greater income than what they can obtain with the value of p derived from equation **(3)**.

Why is it then that, for want of an understanding, the producers do not stop, as in the case of a monopoly or of an association, at the value of p derived from equation **(4)**, which would really give them the greatest income?

The reason is that, producer (1) having fixed his production at what it should be according to equation **(4)** and the condition $D_1 = D_2$, the other will be able to fix his own production at a higher or lower rate with a *temporary benefit*. To be sure, he will soon be punished for his mistake, because he will force the first producer to adopt a new scale of production which will react unfavourably on producer (2) himself. But these successive reactions, far from bringing both producers nearer to the original condition [of monopoly], will separate them further and further from it. In other words, this condition is not one of stable equilibrium; and, although the most favourable for both producers, it can only be maintained by means of a formal engagement; for in the moral sphere men cannot be supposed to be free from error and lack of forethought any more than in the physical world bodies can be considered perfectly rigid, or supports perfectly solid, etc.

45. The root of equation **(3)** is graphically determined by the intersection of the line $y = 2x$ with the curve $y = -\frac{F(x)}{F'(x)}$; while that of equation **(4)** is graphically shown by the intersection of the same curve with the line $y = x$. But, if it is possible to assign a real and positive value to the function $y = -\frac{F(x)}{F'(x)}$ for every real and positive value of x, then the abscissa x of the first point of intersection will be smaller than that of the second, as is sufficiently proved simply by the plot of Fig. 4.

Figure 4



It is easily proved also that the condition for this result is always realized by the very nature of the law of demand. In consequence the root of equation **(3)** is always smaller than that of equation **(4)**; or (as every one believes without any analysis) the result of competition is to reduce prices.

In consequence the root of equation (3) is always smaller than that of equation (4); or the result of competition is to reduce prices.

46. If there were 3, 4, ..., n producers in competition, all their conditions being the same, equation **(3)** would be successively replaced by the following:

$$D + 3p\frac{dD}{dp} = 0, D + 4p\frac{dD}{dp} = 0, \ldots D + np\frac{dD}{dp} = 0;$$

and the value of p which results would diminish indefinitely with the indefinite increase of the number n.

In all the preceding, the supposition has been that natural limitation of their productive powers has not prevented producers from choosing each the most advantageous rate of production. Let us now admit, besides the n producers, who are in this condition, that there are others who reach the limit of their productive capacity, and that the total production of this class is Δ; we shall continue to have the n equations

$$\textbf{(5)} \qquad \begin{cases} f(D) + D_1 f'(D) = 0, \\ f(D) + D_2 f'(D) = 0, \\ \dots \\ f(D) + D_n f'(D) = 0, \end{cases}$$

which will give $D_1 = D_2 = \dots = D_n$, and by addition,

$$nf(D) + nD_1 f'(D) = 0.$$

But $D = nD_1 + \Delta$, whence

$$nf(D) + (D - \Delta)f'(D) = 0,$$

or

$$D - \Delta + np\frac{dD}{dp} = 0.$$

This last equation will now replace equation **(3)** and determine the value of p and consequently of D.

47. Each producer being subject to a cost of production expressed by the functions $\phi_1(D_1), \phi_2(D_2), \dots, \phi_n(D_n)$, the equations of **(5)** will become

$$\textbf{(6)} \qquad \begin{cases} f(D) + D_1 f'(D) - \phi_1'(D_1) = 0, \\ f(D) + D_2 f'(D) - \phi_2'(D_2) = 0, \\ \dots \\ f(D) + D_n f'(D) - \phi_n'(D_n) = 0. \end{cases}$$

If any two of these equations are combined by subtraction, for instance if the second is subtracted from the first, we shall obtain

$$D_1 - D_2 = \frac{1}{f'(D)}[\phi_1'(D_1) - \phi_2'(D_2)]$$

$$= \frac{dD}{dp}[\phi_1'(D_1) - \phi_2'(D_2)].$$

As $\frac{dD}{dp}$ is essentially negative, we shall therefore have at the same time

$$D_1 \gtrless D_2, \text{ and } \phi_1'(D_1) \lessgtr \phi_2'(D_2).$$

Thus the production of plant A will be greater than that of plant B, whenever it will require greater expense to increase the production of B than to increase the production of A by the same amount.

For a concrete example, let us imagine the case of a number of coal mines supplying the same market in competition one with another, and that, in a state of stable equilibrium, mine A markets annually 20,000 hectoliters and mine B, 15,000. We can be sure that a greater addition to the cost would be necessary to produce and bring to market from mine B an additional 1000 hectoliters than to produce the same increase of 1000 hectoliters in the yield of mine A.

This does not make it impossible that the costs at mine A should exceed those at mine B at a lower limit of production. For instance, if the production of each were reduced to 10,000 hectoliters, the costs of production at B might be smaller than A.

48. By addition of equations **(6)**, we obtain

$$nf(D) + Df'(D) - \Sigma\phi_n'(D_n) = 0,$$

or **(7)**

$$D + \frac{dD}{dp}[np - \Sigma\phi_n'(D_n)] = 0.$$

If we compare this equation with the one which would determine the value of p in case all the plants were dependent on a monopolist, viz.

(8)

$$D + \frac{dD}{dp}[p - \phi'(D)] = 0,$$

we shall recognize that on the one hand substitution of the term np for the term p tends to diminish the value of p; but on the other hand substitution of the term $\Sigma\phi_n'(D_n)$ for the term $\phi'(D)$ tends to increase it, for the reason that we shall always have

$$\Sigma\phi_n'(D_n) > \phi'(D);$$

and, in fact, not only is the sum of the terms $\phi_n'(D_n)$ greater than $\phi'(D)$, but even the average of these terms is greater than $\phi'(D)$, *i.e.* we shall have the inequality

$$\frac{\Sigma\phi_n'(D_n)}{n} > \phi'(D).$$

To satisfy one's self of this, it is only necessary to consider that any capitalist, holding a monopoly of productive property, would operate by preference the plants of which the operation is the least costly, leaving the others idle if necessary; while the least favoured competitor will not make up his mind to close his works so long as he can obtain any profit from them, however modest.

Consequently, for a given value of p, or for the same total production, the costs will always be greater for competing producers than they would be under a monopoly.

It now remains to be proved that the value of p derived from equation **(8)** is always greater than the value of p derived from equation **(7)**.

For this we can see at once that if in the expression $\phi'(D)$ we substitute the value of $D = F(p)$, we can change $\phi'(D)$ into a function $\Psi(p)$; and each of the terms which enter into the summational expression $\Sigma\phi_n'(D_n)$, can also be regarded as an implicit function of p, in virtue of the relation $D = F(p)$ and of the system of equations **(6)**. In consequence the root of equation **(7)** will be the abscissa of the point of intersection of the curve

$$(a) \qquad y = -\frac{F(x)}{F(x)},$$

with the curve

$$(b) \qquad y = nx - [\Psi_1(x) + \Psi_2(x) + \ldots + \Psi_n(x)];$$

while the root of equation **(8)** will be the abscissa of the point of intersection of the curve (a) with one which has for its equation

$$(b') \qquad y = x - \Psi(x).$$

As has been already noted, equation (a) is represented by the curve MN (Fig. 5), of which the ordinates are always real and positive; we can represent equation (b) by the curve PQ, and equation (b') by the curve $P'Q'$.

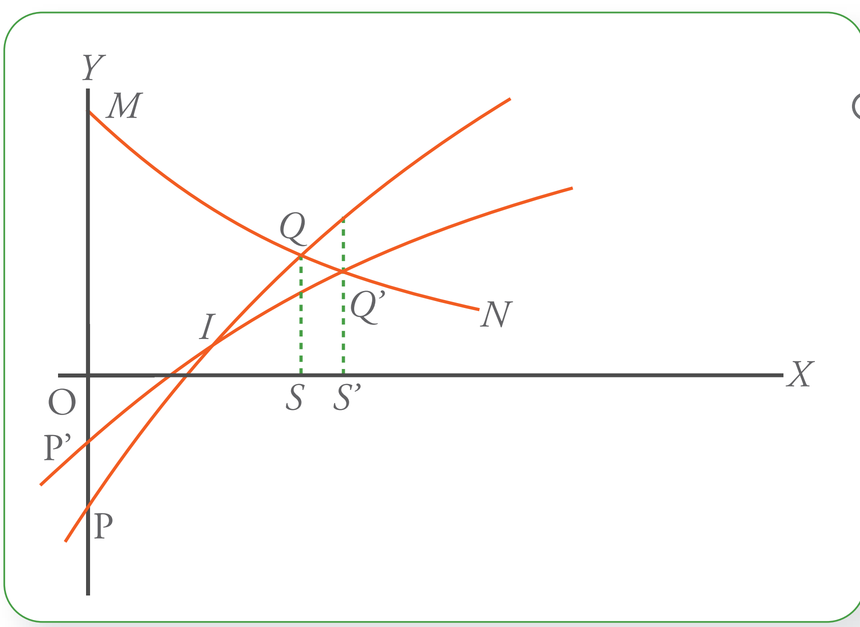


Figure 5

In consequence of the relation just proved, viz.,

$$\Sigma\Psi_n(x) > \Psi(x),$$

we find for the value $x = 0$, $OP > OP'$. It remains to be proved that the curve $P'Q'$ cuts the curve PQ at a point I situated below MN, so that the abscissa of the point Q' will be greater than that of the point Q.

This amounts to proving that at the points Q and Q', the ordinate of the curve (b) is greater than the ordinate of the curve (b') corresponding to the same abscissa.

Suppose that it were not so, and that we should have

$$x - \Psi(x) > nx - [\Psi_1(x) + \Psi_2(x) + \ldots + \Psi_n(x)],$$

or
$$(n-1)x < \Psi_1(x) + \Psi_2(x) + \ldots + \Psi_n(x) - \Psi(x).$$

$\Psi(x)$ is an intermediate quantity between the greatest and smallest of the terms $\Psi_1(x), \Psi_2(x), \ldots, \Psi_{n-1}(x), \Psi_n(x)$; if we suppose that $\Psi_n(x)$ denotes the smallest term of this series, the preceding inequality will involve the following inequality:

$$(n-1)x < \Psi_1(x) + \Psi_2(x) + \ldots + \Psi_{n-1}(x).$$

Therefore x will be smaller than the average of $n - 1$ terms of which the sum forms the second member of the inequality; and among these terms there will be some which are greater than x. But this is impossible, because producer (k), for instance, will stop producing as soon as p becomes less than $\phi_k'(D_k)$ or $\Psi_k(p)$.

49. Therefore if it should happen that the value of p derived from equations **(6)**, combined with the relations

(9) $$D_1 + D_2 + \ldots + D_n = D, \text{ and } D = F(p),$$

should involve the inequality

$$p - \phi_k'(D_k) < 0,$$

it would be necessary to remove the equation

$$f(D) + D_k f'(D) - \phi_k'(D_k) = 0$$

from the list of equations **(6)**, and to substitute for it

$$p - \phi_k'(D_k) = 0,$$

which would determine D_k as a function of p. The remaining equations of **(6)**, combined with equations **(9)**, will determine all the other unknown quantities of the problem.

Chapter IX: Of the Mutual Relations of Producers

55. Very few commodities are consumed in just the form in which they left the hands of the first producer. Ordinarily the same raw material enters into the manufacture of several different products, which are more directly adapted to consumption; and reciprocally several raw materials are generally brought together in the manufacture of each of these products. It is evident that each producer of raw materials must try to obtain the greatest possible profit from his business. Hence it is necessary to inquire according to what laws the profits, which are made by all the producers as a whole, are distributed among the individuals in consequence of the law of consumption for final products. This short summary will suffice to make known what we mean by the influence of the *mutual relations* of producers of different articles, an influence which must not be confounded with that of the *competition* of producers of the same article, which has been analyzed in the preceding chapters.

THIS SHORT SUMMARY WILL SUFFICE TO MAKE KNOWN WHAT WE MEAN BY THE INFLUENCE OF THE *MUTUAL RELATIONS* OF PRODUCERS OF DIFFERENT ARTICLES, AN INFLUENCE WHICH MUST NOT BE CONFOUNDED WITH THAT OF THE *COMPETITION* OF PRODUCERS OF THE SAME ARTICLE.

To proceed systematically, from the simple to the complex, we will imagine two commodities, (*a*) and (*b*), which have no other use beyond that of being jointly consumed in the production of the composite commodity (*ab*); to begin with, we will omit from consideration the expenses caused by the production of each of these raw materials taken separately, and of the costs of making them effective, or of the formation of the composite commodity.

Simply for convenience of expression we can take for examples copper, zinc, and brass under the fictitious hypothesis that copper and zinc have no other use than that of being jointly used to form brass by their alloy, and that the cost of production of copper and zinc can be neglected, as well as the cost of making the alloy.

Let p be the price of a kilogram of brass, p_1 that of a kilogram of copper, and p_2 that of a kilogram of zinc; and m_1:m_2 the proportion of copper to zinc in the brass, so that we should have, according to hypothesis,

(*a*) $$m_1p_1 + m_2p_2 = p.$$

In general, let p, p_1, and p_2 denote the price of the unit of the commodity for the composite article (*ab*) and for the component commodities (*a*) and (*b*); and m_1 and m_2 the numbers of units, or of fractions of the unit, of each component commodity which enter into the formation of the unit of the composite commodity.

Furthermore, let

$$D = F(p) = F(m_1p_1 + m_2p_2)$$

be the demand for the composite commodity, and

$$(b) \qquad \begin{cases} D_1 = m_1F(m_1p_1 + m_2p_2), \\ D_2 = m_2F(m_1p_1 + m_2p_2), \end{cases}$$

the demand for each of the component commodities; if we suppose each of these to be handled by a monopolist, and if we apply to the theory of the mutual relations of producers the same method of reasoning which served for analyzing the effects of competition, we shall recognize that the values of p_1 and p_2 are determined by the two equations

$$\frac{d(p_1D_1)}{dp_1} = 0, \text{ and } \frac{d(p_2D_2)}{dp_2} = 0,$$

of which the development gives

$$\begin{cases} F(m_1p_1 + m_2p_2) + m_1p_1F'(m_1p_1 + m_2p_2) = 0, & (1) \\ F(m_1p_1 + m_2p_2) + m_2p_2F'(m_1p_1 + m_2p_2) = 0; & (2) \end{cases}$$

no other system of values but the one resulting from these equations being compatible with a state of stable equilibrium.

56. To prove this proposition, it is sufficient to show that the curves m_1n_1 and m_2n_2 (which would be the plots of equations (1) and (2), under the hypothesis that the variables p_1 and p_2 represent rectangular coördinates) assume one or the other of the dispositions shown by Figs. 7 and 8; for, if that is admitted, we can show, as in Chapter VII, and, by the same construction, sufficiently indicated by the dotted lines of either figure, that the coördinates of the point of intersection i (or the roots of equations (1) and (2)) are the only values of p_1 and p_2 compatible with stable equilibrium.

We observe that when p_2 is equal to zero, p_1 has a finite value Om_1, *i.e.* the one which renders the product $p_1F(m_1\,p_1)$ a maximum. Thereupon, as p_2 increases, the value of p_1, which will procure the greatest profit for producer (1), may continue to increase (as is the case in Fig. 7), or to decrease (as is the case in Fig. 8); but, even under the latter hypothesis, it can never become absolutely equal to zero. The one case or the other will occur according to the form of the function F, and according as we find

$$\frac{[F'(p)]^2 - F(p) \cdot F''(p)}{2\,[F'(p)]^2 - F(p) \cdot F''(p)} \gtrless 0.$$

In this inequality p denotes a function of p_1 and p_2, determined by equation (a).

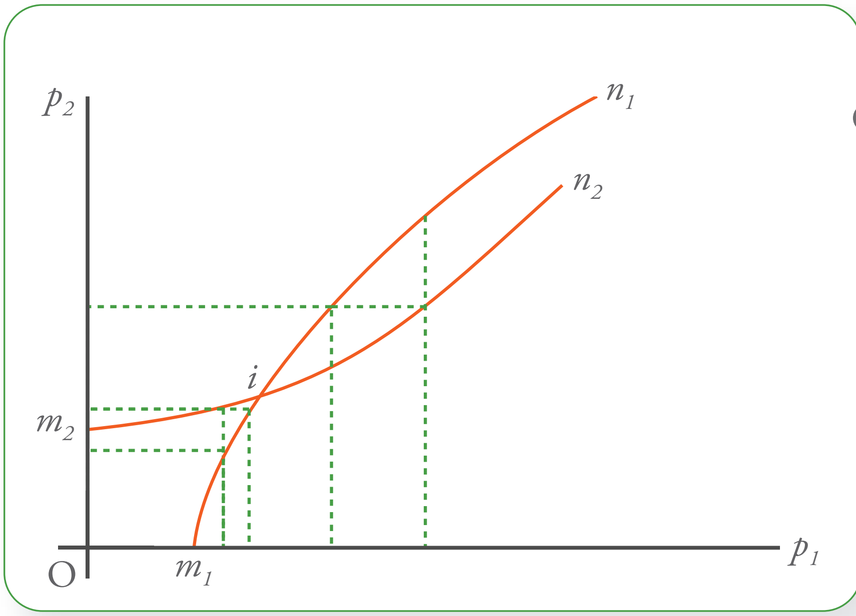


Figure 7

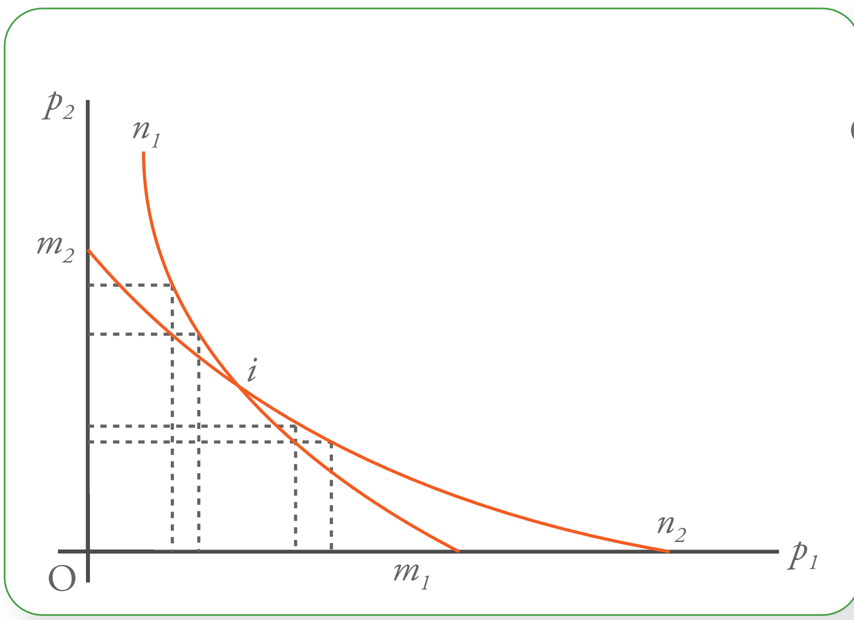


Figure 8

But since equations **(1)** and **(2)** and the preceding inequality are symmetrical with reference to m_1p_1 and m_2p_2, it will result that, whenever the form of the function F is such that the ordinates p_2 of the curve m_1n_1 continue to increase for increasing values of p_1, then the abscissas p_1 of the curve m_2n_2 will go on increasing for increasing values of p_2, so that the two curves will assume the disposition represented by Fig. 7. On the contrary, whenever the ordinates p_2 of the curve m_1n_1 decrease for increasing values of p_1, the abscissas p_1 of the curve m_2n_2 will likewise go on decreasing for increasing values of p_2, and then the two curves will assume the disposition represented by Fig. 8.

57. As equations **(1)** and **(2)** can be considered as determined, in consequence of the previous discussion, we will remark that they yield at once

$$m_1 p_1 = m_2 p_2 = \frac{1}{2} p;$$

that is to say, that by the purely abstract hypothesis under consideration, the profits would be equally divided between the two monopolists; and, in fact, there would be no reason why the division should be unequal, and to the profit of one rather than of the other.

By addition of equations **(1)** and **(2)**, we can deduce

$$(c) \qquad F(p) + \frac{1}{2} pF'(p) = 0,$$

while, if the interests of the two producers had remained undistinguished, p would have been determined by the condition that $pF(p)$ should be a maximum, *i.e.* by the equation

$$(c') \qquad F(p) + pF'(p) = 0.$$

To prove the accuracy of this distinction, exactly the same method of reasoning should be used that we took in treating of the competition of producers.

But there is this essential and very remarkable difference, that the root of equation (c) is always greater than that of equation (c'), so that the composite commodity will always be made more expensive, by reason of separation of interests than by reason of the fusion of monopolies. An association of monopolists, working for their own interest, in this instance will also work for the interest of consumers, which is exactly the opposite of what happens with competing producers.

An association of monopolists, working for their own interest, in this instance will also work for the interest of consumers, which is exactly the opposite of what happens with competing producers.

Furthermore, the higher value of the root of equation (c) than of that of equation (c') can be shown by the same graphical construction which served to establish the opposite result in the chapter in which we treated of competition.

If we had supposed n commodities thus related, instead of only two, equation (c) would evidently have been replaced by

$$F(p) + \frac{1}{n} pF'(p) = 0;$$

from which we should conclude, that the more there are of articles thus related, the higher the price determined by the division of monopolies will be, than that which would result from the fusion or associations of the monopolists.

58. Such a form might be given to the function F that the curves represented by equations **(1)** and **(2)** would not intersect; for instance, if it were

$$F(p) = \frac{a}{b + p^2},$$

equations **(1)** and **(2)** would become

$$b - m_1^2 p_1^2 + m_2^2 p_2^2 = 0, \text{ and } b + m_1^2 p_1^2 + m_2^2 p_2^2 = 0,$$

and would represent two conjugate hyperbolas (Fig. 9), of which the limbs $m_1 n_1$ and $m_2 n_2$ have a common asymptote and cannot meet.

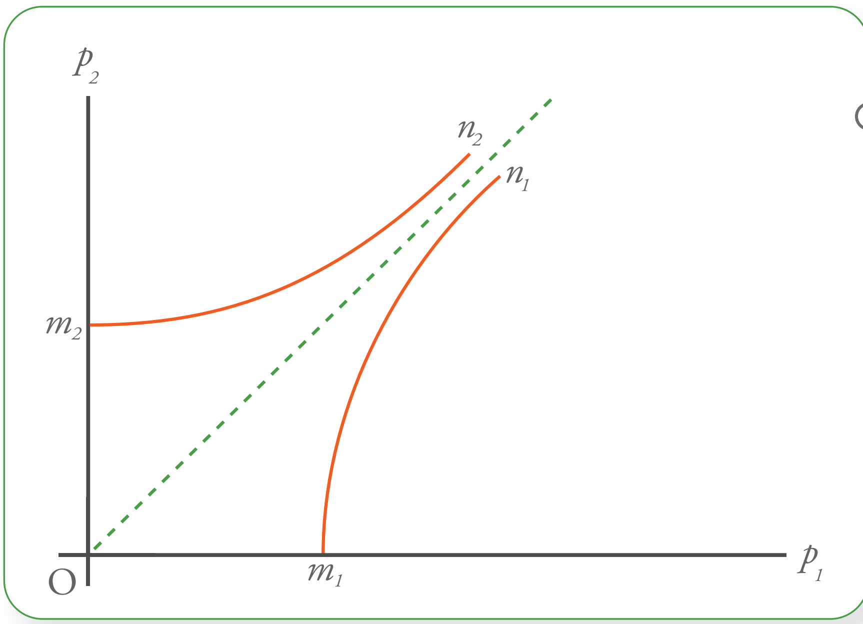


Figure 9

A passing note is sufficient for these peculiarities of analysis, which cannot have any application to actual events.

Another peculiarity of the same kind would appear if we suppose that the roots of equations **(1)** and **(2)** establish a value of p, and, consequently, a value of D which exceeds the quantity which one or other of the producers can furnish. Let Δ be the limit which D cannot exceed, because of a necessary limitation in the production of one or the component articles, and π the corresponding limit of p according to the relation $D = F(p)$. We shall therefore have

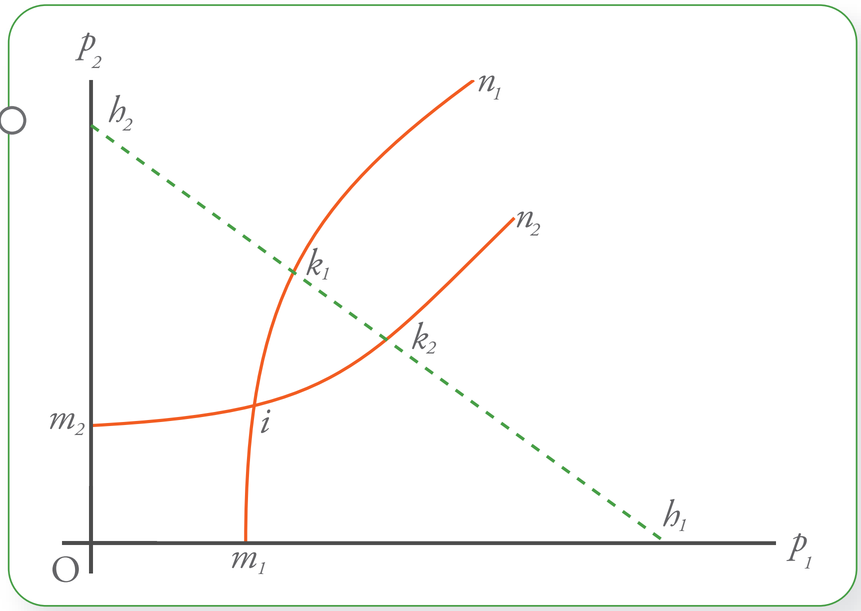
$$m_1 p_1 + m_2 p_2 > \pi;$$

i.e. the variables p_1 and p_2 can be the coördinates only of a point situated above the line h_1h_2 (Fig. 10), which would have for its equation

$$m_1p_1 + m_2p_2 = \pi;$$

and consequently, if the point i, where the two curves m_1n_1 and m_2n_2 intersect, falls below the line h_1h_2, its coördinates cannot be taken for the values of p_1 and p_2.

Figure 10



From this the conclusion can be drawn, if necessary by aid of the graphical construction indicated above, that the values of p_1 and p_2 are indeterminate, being subject only to this condition, that the points which would have the values of these variables for coördinates fall on the part k_1k_2 of the line, which is intercepted between the curves m_1n_1 and m_2n_2.

This singular result springs from an abstract hypothesis of the nature of those which we can discuss in this essay. It is very plain that in the order of actual facts, and where all the conditions of an economic system are accounted for, there is no article of which the price is not completely determined.

59. We will now take into consideration the costs of production of the two component articles, which we will represent by the functions $\phi_1(D_1)$ and $\phi_2(D_2)$. The values of p_1 and p_2 will now result from the two equations

$$(d) \qquad \begin{cases} \dfrac{d[p_1D_1 - \phi_1(D_1)]}{dp_1} = 0, \\ \dfrac{d[p_2D_2 - \phi_2(D_2)]}{dp_2} = 0, \end{cases}$$

which will become, by reason of equations (a) and (b),

$$(e_1) \quad F(m_1p_1 + m_2p_2) + m_1F'(m_1p_1 + m_2p_2) \cdot [p_1 - \phi_1'(D_1)] = 0,$$

$$(e_2) \quad F(m_1p_1 + m_2p_2) + m_2F'(m_1p_1 + m_2p_2) \cdot [p_2 - \phi_2'(D_2)] = 0.$$

From these we derive

$$m_1[p_1 - \phi_1'(D_1)] = m_2[p_2 - \phi_2'(D_2)],$$

or, by reason of the condition

$$\frac{m_1}{m_2} = \frac{D_1}{D_2},$$

$$D_1[p_1 - \phi_1'(D_1)] = D_2[p_2 - \phi_2'(D_2)].$$

From this it follows that if the functions $\phi_1'(D_1)$ and $\phi_2'(D_2)$ reduce to constants, the net profits of the two coöperating producers will be equal. But this will no longer be so in the more general case where the functions $\phi_1'(D_1)$ and $\phi_2'(D_2)$ vary respectively with D_1 and D_2. The net profits of the two producers will then be expressed by

$$D_1\left[p_1 - \frac{\phi_1(D_1)}{D_1}\right] \text{ and } D_2\left[p_2 - \frac{\phi_2(D_2)}{D_2}\right];$$

so that if we have, for instance,

$$\phi_1'(D_1) > \frac{\phi_1(D_1)}{D_1} \text{ and } \phi_2'(D_2) < \frac{\phi_2(D_2)}{D_2},$$

the net profit of producer (1) will be greater than that of producer (2). From equation (a) and equations (e_1) and (e_2) there can further be deduced

$$(f) \qquad 2F(p) + F'(p)[p - m_1\phi_1'(D_1) - m_2\phi_2'(D_2)] = 0,$$

$$m_1p_1 = \frac{1}{2}[p + m_1\phi_1'(D_1) - m_2\phi_2'(D_2)],$$

and

$$m_2p_2 = \frac{1}{2}[p + m_1\phi_1'(D_1) - m_2\phi_2'(D_2)].$$

But if there had been a fusion of monopolies, equation (f) would have been replaced by

$$(f') \qquad F(p) + F'(p)[p - m_1\phi_1'(D_1) - m_2\phi_2'(D_2)] = 0.$$

By recourse to the graphic representation which has served us for similar cases, it will easily be recognized that the root of equation (f) is greater than that of equation (f'), and, therefore, that an increase in price is the result of separation of the monopolies.

60. Up to this point we have neglected to account for the expenses involved in putting the raw materials to use in the formation of the resultant article, as well as the transportation of this resultant commodity to the market where it is consumed, the taxes which may be imposed on it, etc.

But if we suppose that these expenses are proportional to the quantity turned out, which is ordinarily the case, and that the sum of these expenses, for each unit of the resultant article, is expressed by the constant h, equation (a) will be replaced by

$$p = m_1 p_1 + m_2 p_2 + h,$$

and instead of equation (f) we shall have

$$2F(p) + F'(p)[p - h - m_1\phi_1'(D_1) - m_2\phi_2'(D_2)] = 0.$$

Thus the result will be the same as if the expenses had been borne directly by producers (1) and (2), and as if the burden of these expenses had been divided between them in the ratio of m_1 to m_2.

61. By a less restricted hypothesis than the one which we have considered till now, each of the component articles is susceptible of various uses besides that of coöperating in the formation of the composite article. Let $F(p)$ be, as before, the demand for the composite article, and $F_1(p_1)$ and $F_2(p_2)$ the demand for article (1) and that for article (2), for other uses than that of coöperating in the production of the composite article. The values of p_1 and p_2 will still be given by the equations (d), but we shall have

$$D_1 = F_1(p_1) + m_1 F(m_1 p_1 + m_2 p_2),$$

and

$$D_2 = F_2(p_2) + m_2 F(m_1 p_1 + m_2 p_2),$$

by reason of which the equations (d) become

$$F_1(p_1) + m_1 F(m_1 p_1 + m_2 p_2) + [F_1'(p_1) + m_1^2 F'(m_1 p_1 + m_2 p_2)][p_1 - \phi_1'(D_1)] = 0,$$

$$F_2(p_2) + m_2 F(m_1 p_1 + m_2 p_2) + [F_2'(p_2) + m_2^2 F'(m_1 p_1 + m_2 p_2)][p_2 - \phi_2'(D_2)] = 0.$$

These expressions thus become too complicated to make it easy to derive any general consequences from them. Without further delay we will therefore pass on to a case far more important, and which can easily be treated in as general a manner as is desired. This is the case where each of the two articles concurrently used is produced under the influence of unlimited competition.

62. According to the theory developed in Chapter VIII, we now obtain two series of equations:

$$(a_1)\quad \begin{cases} p_1 - \bar{\phi}_1'\bar{D}_1 = 0, \\ p_1 - \phi_2'\bar{D}_2 = 0, \\ \quad \ldots \\ p_1 - \bar{\phi}_n'\bar{D}_n = 0; \end{cases} \qquad (a_2)\quad \begin{cases} p_2 - \bar{\bar{\phi}}_1'\bar{\bar{D}}_1 = 0, \\ p_2 - \bar{\bar{\phi}}_2'\bar{\bar{D}}_2 = 0, \\ \quad \ldots \\ p_2 - \bar{\bar{\phi}}_n'\bar{\bar{D}}_n = 0. \end{cases}$$

Over the letters ϕ and D we set one or two horizontal lines according as they relate to article (1) or article (2). The subscripts to these letters serve to distinguish the producers in each of the two series.

Together with the equations of (a_1) and (a_2) the two following equations should be considered:

$$(b_1)\qquad \bar{D}_1 + \bar{D}_2 + \ldots + \bar{D}_n = F_1(p_1) + m_1F(m_1p_1 + m_2p_2),$$

$$(b_2)\qquad \bar{\bar{D}}_1 + \bar{\bar{D}}_2 + \ldots + \bar{\bar{D}}_n = F_2(p_2) + m_2F(m_1p_1 + m_2p_2).$$

If we deduce from the equations of (a_1) and (a_2) the values of $\bar{D}_1$, $\bar{D}_2 \ldots$ and $\bar{\bar{D}}_1$, $\bar{\bar{D}}_2 \ldots$ as functions of p, equations (b_1) and (b_2) will assume the forms

$$\Omega_1(p_1) = F_1(p_1) + m_1F(m_1p_1 + m_2p_2), \tag{3}$$

$$\Omega_2(p_2) = F_2(p_2) + m_2F(m_1p_1 + m_2p_2), \tag{4}$$

in which $\Omega_1(p_1)$ denotes a function of p_1 which increases with p_1, and $\Omega_2(p_2)$ another function of p_2 which increases with p_2.

Suppose that the production of article (1) is subjected to an increase of expense u, such as would result from a specific tax; the values of p_1 and p_2, which before the increase in expense were determined by equations (3) and (4), will become $p_1 + \delta_1$ and $p_1 + \delta_2$, and we shall have, to determine δ_1 and δ_2, the equations

$$\Omega_1(p_1 + \delta_1 - u) = F_1(p_1 + \delta_1) + m_1F(m_1p_1 + m_2p_2 + m_1\delta_1 + m_2\delta_2), \tag{5}$$

$$\Omega_2(p_2 + \delta_2) = F_2(p_2 + \delta_2) + m_2F(m_1p_1 + m_2p_2 + m_1\delta_1 + m_2\delta_2). \tag{6}$$

If we admit that in comparison with p_1 and p_2, u, δ_1, and δ_2 are small fractions, of which the powers higher than the first can be omitted in our calculations, then equations (5) and (6) will become, in virtue of equations (3) and (4),

$$\delta_1\{\Omega_1'(p_1) - F_1'(p_1) - m_1^2F'(m_1p_1 + m_2p_2)\} - \delta_2m_1m_2F'(m_1p_1 + m_2p_2) = u\Omega_1'(p_1),$$

and

$$-\delta_1m_1m_2F'(m_1p_1 + m_2p_2) + \delta_2\{\Omega_2'(p_2) - F_1'(p_1) - m_2^2F'(m_1p_1 + m_2p_2)\} = 0.$$

To simplify the notation, we will write Ω_1' instead of $\Omega_1'(p_1)$, F' instead of $F'(m_1p_1 + m_2p_2)$, and so on throughout. Finally, let us put

$$Q = \Omega_1'\Omega_2' - \Omega_1'F_2' - \Omega_2'F_1' - m_2^2F'\Omega_1' - m_1^2F'\Omega_2' + F_1'F_2' + m_1^2F'F_2' + m_2^2F'F_2'.$$

From this and from the two preceding equations we can derive

(7) $$\delta_1 = \frac{u}{Q} \cdot (\Omega_1'\Omega_2' - \Omega_1'F_2' - m_2^2F'\Omega_1'),$$

and **(8)** $$\delta_2 = \frac{u}{Q} \cdot m_1m_2\Omega_1'F'.$$

If we observe that the quantities Ω_1' and Ω_2' are essentially positive, whereas the quantities F', F_1', and F_2' are essentially negative, inspection of the values of δ_1 and δ_2 will now permit us to observe the following results:

1. δ_1 is of the same sign as u; for $\frac{\delta_1}{u}$ is equal to a fraction, of which both numerator and denominator have all their terms positive.

2. δ_1 is smaller than u; for the denominator of the aforementioned fraction contains all the terms of the numerator, and besides them a number of terms which are all positive.

3. δ_2 is of opposite sign to δ_1; for the denominator of the fraction $\frac{\delta_1}{u}$ is the same as that of the fraction $\frac{\delta_2}{u}$, and the numerator of this latter fraction is a negative quantity.

Although we only obtained these results by supposing u, δ_1 and δ_2 very small with reference to p_1 and p_2, it is easy to see that this restriction can be removed by supposing that any increase of expense, of whatever kind, takes place by a succession of very small increments. As the signs of the quantities Ω' and F' do not change in the passage from one state to the other, the relations which we have just found between the elementary variations u, δ_1, and δ_2 will also hold between the sums of these elements (Article 32).

There must also result a smaller consumption or production of article (2); and, as this article is not subjected to an increase in the cost of production, the restriction of production for this article can only be caused by a decrease in the price.

In consequence, any increase in expense in the production of article (1) will increase the price of that article, but, nevertheless, so that the rise is less than the increase in expense; and at the same time the price of article (2) will fall.

It would be easy to show the necessity of all these results by methods of reasoning, independent of the preceding calculations. If article (1) did not rise in price when affected by an increase in cost, the producers of it would be obliged to restrict their output to avoid a loss, and it is impossible that the price should fail to increase when the quantity delivered diminishes. The article must rise therefore, and must rise less than the increase in cost, as otherwise the producers would have no reason for restricting their output. Finally, since there results a smaller consumption of article (1), as well for the manufacture of the composite article as for all other uses, there must also result a smaller consumption or production of article (2); and, as this article is not subjected to an increase in the cost of production, the restriction of production for this article can only be caused by a decrease in the price.

The variation in the price of the composite article, resulting from the opposite variations δ_1 and δ_2 in the prices of the component articles, is equal to $m_1\delta_1 + m_2\delta_2$, and from equations **(7)** and **(8)** we obtain

$$m_1\delta_1 + m_2\delta_2 = m_1 u \cdot \frac{\Omega_1'(\Omega_2' - F_2')}{Q}.$$

It results from this expression that the variation in the price of the composite article is of the same sign as u and δ_1, and that it is less than $m_1 u$ which is as it should be, on account of the fall in the price of article (2).

If we suppose any number of articles used concurrently, it could be demonstrated in the same manner, and by calculations which would offer no other difficulty than their length, (1) that an increase in cost occurring in the production of one of the articles, raises the price of this article and that of the composite article, and causes a fall in the prices of all the other component articles; (2) that the increase in the price of the article affected is less than the increase in cost or than the tax laid upon it.

63. Let us now consider the case where the increase in cost u falls directly on the composite article, whether it is a specific tax imposed on this article, on an increase occurring in the cost of distribution of the article to consumers. Equations **(3)** and **(4)** will be replaced by

$$\Omega_1(p_1 + \delta_1) = F_1(p_1 + \delta_1) + m_1 F(m_1 p_1 + m_2 p_2 + m_1\delta_1 + m_2\delta_2 + u),$$

and

$$\Omega_2(p_2 + \delta_2) = F_2(p_2 + \delta_2) + m_2 F(m_1 p_1 + m_2 p_2 + m_1\delta_1 + m_2\delta_2 + u);$$

and these, when treated as were equations **(5)** and **(6)**, will give

$$\delta_1\Omega_1' = \delta_1 F_1' + m_1^2\delta_1 F' + m_1 m_2\delta_2 F' + m_1 u F',$$

and
$$\delta_2\Omega_2' = \delta_2 F_2' + m_1 m_2\delta_1 F' + m_2^2\delta_2 F' + m_2 u F';$$

from which we derive

$$\delta_1 = \frac{um_1F'(\Omega_2' - F_2')}{Q},$$

and
$$\delta_2 = \frac{um_2F'(\Omega_1' - F_1')}{Q},$$

in which the polynomial represented by Q is composed of the same terms as in the preceding article.

From these expressions we easily conclude, in virtue of the signs of the quantities Ω' and F':

1. That both δ_1 and δ_2 are of the opposite sign to u.
2. That the quantity $m_1\delta_1 + m_2\delta_2$ is numerically less than u.

Moreover, the variations δ_1 and δ_2 in the prices of the component articles are mutually connected by this vary simple relation:

$$\frac{\delta_1}{\delta_2} = \frac{m_1(\Omega_2' - F_2')}{m_2(\Omega_1' - F_1')},$$

which is independent of the function F. Consequently, any increase of expense, or any tax which affects the composite article, will lower the prices of the component commodities, and at the same time will raise the price of the composite article, but by a quantity less than u, since this rise in price will be expressed by

$$u = m_1\delta_1 + m_2\delta_2,$$

and since $m_1\delta_1 + m_2\delta_2$ is, as we have just seen, numerically less than u, and of opposite sign.

These results can readily be generalized, whatever the number and kind of the component commodities, so long as they are produced under the influence of unlimited competition. They are worthy of serious consideration, as they have all the certainty of mathematical theorems, without being such as must, on that account, be excluded from the number of practical truths.

64. Let us go on to the case where article (2) has a limit to its production, so that the value of p_2 derived from equations **(3)** and **(4)** would correspond to a demand for this article which its producers could not satisfy. If we denote by Δ_2 this limit of production, the values of p_1 and p_2 will be determined by the system of equations

$$\Omega_1(p_1) = F_1(p_1) + m_1F(m_1p_1 + m_2p_2),$$

and
$$\Delta_2 = F_2(p_2) + m_2F(m_1p_1 + m_2p_2).$$

Under these circumstances there will be no change in the equations which determine the values of p_1 and p_2, if we suppose that there falls on article (2) a tax, or an increase in the cost of production, denoted by u; consequently these values will remain the same, and the entire increase in the cost will be borne by producers of (2), without any loss resulting to the consumers of the component commodities, or of the composite article.

THE ENTIRE INCREASE IN THE COST WILL BE BORNE BY PRODUCERS OF (2), WITHOUT ANY LOSS RESULTING TO THE CONSUMERS OF THE COMPONENT COMMODITIES, OR OF THE COMPOSITE ARTICLE.

If the tax u falls on article (1), both of the old prices p_1 and p_2 will vary, and may be represented by $p_1 + \delta_1$ and $p_2 + \delta_2$. Equations **(5)** and **(6)** are applicable to this case by replacing the function $\Omega_2(p_2 + \delta_2)$ in the second of these equations by the constant Δ_2, which amounts to supposing the derivative Ω_2' equal to zero in the formulas derived from these equations.

Thus, under the hypothesis that the variations u, δ_1, and δ_2 can be treated as very small quantities, we shall have:

$$\delta_1 = \frac{-u\Omega_1'(F_2' + m_2^2F')}{R},$$

and

$$\delta_2 = \frac{um_1m_2\Omega_1'F'}{R},$$

$$\frac{\delta_1}{\delta_2} = -\frac{F_2' + m_2^2F'}{m_1m_2F'},$$

$$m_1\delta_1 + m_2\delta_2 = \frac{-um_1\Omega_1'F_2'}{R};$$

in which the composition of the polynomial R is given by the auxiliary equation

$$R = -\Omega_1'(F_2' + m_2^2F') + F_1'F_2' + m_1^2F'F_2' + m_2^2F'F_1'.$$

From these equations are derived the following consequences, which are applicable to all values of the variations u, δ_1, and δ_2:

1. δ_1 is of the same sign as u, and numerically smaller; the article affected by the tax increases in price, but by an amount less than the tax, so that there will be a diminution in the quantity produced and in the income of its producers;

2. δ_2 is of opposite sign to u, so that the article which is not directly affected by the tax falls in price, to the disadvantage of the producers of this article, even though the quantity produced does not vary;

3. $m_1\delta_1 + m_2\delta_2$ is of the same sign as u; thus the composite article will rise in price, the rise of the taxed article more than compensating for the fall of the other article.

It would be found in the same way that the prices of both component articles would fall if the tax or the increase in cost bears directly on the resultant article.

65. Let us now suppose that for some reason the limit Δ_2 changes and becomes $\Delta_2 + v_2$ without the occurrence of any change in the cost of production. Treating, according to our method, the variation v_2 and the resulting variations δ_1 and δ_2 to begin with as very small, we shall have:

$$\delta_1 = v_2 \cdot \frac{-m_1 m_2 F'}{R},$$

$$\delta_2 = v_2 \cdot \frac{-(\Omega_1' - F_1' - m_1^2 F')}{R},$$

$$m_1\delta_1 + m_2\delta_2 = v_2 \cdot \frac{-m_2(\Omega_1' - F_1')}{R}.$$

From these expressions we conclude that whatever the extent of the variations, raising the limit Δ_2 depresses the price of article (2), and raises the price of article (1), but in a less degree, so that it brings about a fall in the price of the resultant article. ◀